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C^* -ALGEBRA GENERATED BY THE PATH SEMIGROUP

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ABSTRACT. In this paper we study the structure of the C^* -algebra, generated by the representation of the path semigroup on a partially ordered set (poset) and get a net of isomorphic C^* -algebras over this poset. We construct the extensions of this algebra, such that the algebra is an ideal in that extensions and quotient algebras are isomorphic to the Cuntz algebra.

1. INTRODUCTION

In the algebraic approach to the quantum field theory [1] (the algebraic quantum field theory) the physical content of the theory is encoding by a collection of C^* -algebras of observables $\mathcal{A} = \{\mathcal{A}_o\}_{o \in K}$ indexed by elements of a partially ordered set K (poset) [2]. The poset K is a non-empty set with a binary relation \leq which is reflexive, antisymmetric and transitive. A net of C^* -algebras over the poset K is the pair $(\mathcal{A}, \gamma)_K$, where $\gamma = \{\gamma_{o'o} : \mathcal{A}_o \rightarrow \mathcal{A}_{o'}\}_{o \leq o'}$ are $*$ -morphisms fulfilling the net relations

$$\gamma_{o''o} = \gamma_{o''o'} \circ \gamma_{o'o}$$

for all $o \leq o' \leq o'' \in K$. If we consider the poset K as a category in which objects are elements of K and morphisms are arrows (o, o') for all $o \leq o' \in K$, then the net of C^* -algebras represents a covariant functor from a poset K to category of unital C^* -algebras with $*$ -morphisms (see for example [3, 4]). More precisely we have a net of C^* -algebras for an

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upward directed poset and in the event of non-upward directed we obtain a precosheaf of C^* -algebras [5–7].

In this paper we give an algebraic notion of a path on a poset K which turns out to be relevant to the point of view on a path as a sequence of 1-simplices. We introduce the path semigroup S on the given poset K and construct a new C^* -algebra $C_{red}^*(S)$ generated by the representation of S . We consider both an upward directed set K and non-upward directed. The present paper is addressed to detailed study of the path semigroup S and the C^* -algebra $C_{red}^*(S)$. We construct the net of isomorphic C^* -algebras $\{\mathcal{A}_a, \gamma_{ba}, a \leq b\}_{a,b \in K}$ over the poset K , where \mathcal{A}_a are restrictions of the algebra $C_{red}^*(S)$ on Hilbert subspaces and $\gamma_{ba} : \mathcal{A}_a \rightarrow \mathcal{A}_b$ are $*$ -isomorphisms, such that $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for all $a \leq b \leq c \in K$. In the last section we consider extensions $C_{red,n}^*(S)$ and $C_{red,\infty}^*(S)$ of the algebra $C_{red}^*(S)$. We prove that $C_{red}^*(S)$ is an ideal in $C_{red,n}^*(S)$ and also in $C_{red,\infty}^*(S)$. We show that quotient algebras $C_{red,n}^*(S)/C_{red}^*(S)$ and $C_{red,\infty}^*(S)/C_{red}^*(S)$ are isomorphic to the Cuntz algebra.

Several works in recent years have addressed the C^* -algebras generated by the left regular representations of semigroups with reduction [8] and by the representations of an inverse semigroup [9–11]. In the paper [12] have shown that the Cuntz algebra can be represented as a C^* -crossed product by endomorphisms of the CAR algebra.

2. PATH SEMIGROUP

In this section we define the path semigroup S on a partially ordered set K . The semigroup S is an inverse semigroup and has subgroups G_a corresponding to loops which start and end at the same point $a \in K$.

Let K be a partially ordered set with binary relation \leq satisfying reflexivity, antisymmetry and transitivity conditions. We call the set K a *poset*. Elements a and b are called *comparable* on K if $a \leq b$ or $b \leq a$. We say that the poset K is *upward directed* if for every pair $a, b \in K$ there exists $c \in K$, such that $a \leq c$ and $b \leq c$.

We call an ordered pair of comparable elements a and b on K an *elementary path*. We denote it by (b, a) if $b \leq a$ and by $\overline{(b, a)}$ if $b \geq a$ and say that a is a *starting point* of p , and b is an *ending point*. We use the notation $\partial_1 p = a$ to denote the starting point of p and $\partial_0 p = b$ to denote the ending point. For an elementary path $p = (b, a)$ we define the *inverse path* $p^{-1} = \overline{(a, b)}$. For $p = \overline{(b, a)}$ the inverse path is $p^{-1} = (a, b)$. Obviously, $(p^{-1})^{-1} = p$. Finally we call the pair $(a, a) = \overline{(a, a)} = i_a$ a *trivial path*.

Let p_1, \dots, p_n be elementary paths, such that $\partial_0 p_{i-1} = \partial_1 p_i$ for $i = 2, \dots, n$. We define a *path* p as the sequence

$$p = p_n * p_{n-1} * \dots * p_1.$$

The starting point of p is $\partial_1 p = \partial_1 p_1$ and the ending point is $\partial_0 p = \partial_0 p_n$. For every path $p = p_n * p_{n-1} * \dots * p_1$ the inverse path is

$$p^{-1} = p_1^{-1} * p_2^{-1} * \dots * p_n^{-1}$$

with $\partial_1 p^{-1} = \partial_0 p$ and $\partial_0 p^{-1} = \partial_1 p$. Let us consider the set of all paths on K . We denote an *empty path* 0 as a formal symbol. The empty path 0 has neither the starting point nor the ending point. We define a semigroup structure on the set of all paths with the empty path by extending the operation " $*$ " to multiplication as

$$p * q = \begin{cases} p * q & \text{if } p \neq 0, q \neq 0 \text{ and } \partial_1 p = \partial_0 q, \\ 0 & \text{otherwise} \end{cases}$$

for all paths p and q .

The poset K is called *connected* if for all $a, b \in K$ there exists a path p , such that $\partial_0 p = a$, $\partial_1 p = b$. Throughout the rest of this article we assume K be a connected set.

We call the set of all paths on K with the empty path a *path semigroup* and denote it by S if for all $a, b, c \in K$, such that $a \leq b \leq c$, the following axioms hold:

1. $\overline{(a, b)} * \overline{(b, c)} = \overline{(a, c)}$;
2. $\overline{(c, b)} * \overline{(b, a)} = \overline{(c, a)}$;
3. $\overline{(b, a)} * (a, b) = i_b$, $(a, b) * \overline{(b, a)} = i_a$;
4. $\overline{(a, b)} * i_b = \overline{(a, b)}$, $i_a * (a, b) = \overline{(a, b)}$;
5. $\overline{(b, a)} * i_a = \overline{(b, a)}$, $i_b * \overline{(b, a)} = \overline{(b, a)}$;
6. $i_a * i_a = i_a$.

It is easy to see that path semigroup S has the following useful properties:

- 1) for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$,

$$p^{-1} * p = i_b, \quad p * p^{-1} = i_a;$$

- 2) for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$,

$$i_a * p = p * i_b = p;$$

- 3) for all $p, q \in S$, such that $\partial_0 q = \partial_1 p$,

$$(p * q)^{-1} = q^{-1} * p^{-1};$$

4) for all $p, q, s \in S$ if $p * q = p * s \neq 0$ or $q * p = s * p \neq 0$ then $q = s$;
so the path semigroup S is a semigroup with a reduction.

Thus, we can write elements of S as follows:

$$(1) \quad p = (a_{2n}, a_{2n-1}) * \dots * \overline{(a_3, a_2)} * (a_2, a_1) * \overline{(a_1, a_0)}.$$

Here elementary paths of type (a, b) and $\overline{(a, b)}$ alternate with each other. Note that there exists a variety of representations of type (1) for a path p . Our definition of the path turns out to be in full accordance with the definition given in [4]. The multiplication $(a_{i+1}, a_i) * \overline{(a_i, a_{i-1})}$ is 1-simplex with support a_i where elements a_{i-1}, a_i, a_{i+1} are 0-simplices (see definitions of 0-simplex and 1-simplex in [2–4]).

Three elements $a, c, x \in K$, such that $a, c \leq x$, form a 1-simplex denoted by

$$[a^x c] = (a, x) * \overline{(x, c)}$$

with support x . An inverse 1-simplex is

$$[c^x a] = (c, x) * \overline{(x, a)}$$

with the same support. In general a 1-simplex depends on the support. But for example if $x, y \in K$ are comparable elements then

$$(2) \quad [a^x c] = [a^y c].$$

Indeed for $x \leq y$ we observe

$$[a^y c] = (a, y) * \overline{(y, c)} = (a, x) * (x, y) * \overline{(y, x)} * \overline{(x, c)} = (a, x) * i_x * \overline{(x, c)} = [a^x c].$$

In Lemma 4 we show that a 1-simplex does not depend from the support if the poset is upward directed.

Therefore, one can rewrite the path (1) as a sequence of 1-simplices:

$$p = [a_{2n}^{a_{2n-1}} a_{2n-2}] * \dots * [a_2^{a_1} a_0].$$

Let us recall the definition of an inverse semigroup (for details see [13–15]). Let S be a semigroup. Elements $a, b \in S$ are called *mutual inverses* if

$$a = aba, \quad b = bab.$$

The semigroup S is called an *inverse semigroup* if for every $a \in S$ there exists a unique inverse element $b \in S$.

We use the following theorem in the proof of Lemma 1.

Theorem 1 ([15]). *For a semigroup S in which every element has an inverse, uniqueness of inverses is equivalent to the requirement that all idempotents in S commute.*

Lemma 1. *The path semigroup S is an inverse semigroup.*

Proof. Let $p \in S$ be a path with a starting point $\partial_1 p = a$ and an ending point $\partial_0 p = b$. For every p there is an inverse path p^{-1} , such that

$$p * p^{-1} * p = i_b * p = p, \quad p^{-1} * p * p^{-1} = i_a * p^{-1} = p^{-1}.$$

Hence, p and p^{-1} are mutual inverses elements. For every $a \in K$ we have $i_a * i_a = i_a$ and $i_a * i_b = 0$ for all $a \neq b$. Therefore the set $\{i_a\}_{a \in K}$ forms a commutative subsemigroup of idempotents in the path semigroup S . Hence, by Theorem 1 the path semigroup S is an inverse semigroup. \square

Lemma 2. *If for some 1-simplices $[a^x b]$ and $[b^y c]$ there exists $z \in K$, such that $x, y \leq z$, then $[a^x b] * [b^y c] = [a^z c]$.*

Proof. We have

$$\begin{aligned} [a^x b] * [b^y c] &= (a, x) * \overline{(x, b)} * (b, y) * \overline{(y, c)} \\ &= (a, x) * (x, z) * \overline{(z, x)} * \overline{(x, b)} * (b, y) * (y, z) * \overline{(z, y)} * \overline{(y, c)} \\ &= (a, z) * \overline{(z, b)} * (b, z) * \overline{(z, c)} = (a, z) * \overline{(z, c)} = [a^z c]. \end{aligned}$$

\square

Corollary 1. *If for some 1-simplices $[a^x b]$, $[b^y c]$ and $[a^z c]$ there exists $w \in K$, such that $x, y, z \leq w$, then $[a^x b] * [b^y c] = [a^z c]$.*

Proof. Using the Lemma 2 and the equality (2) we have $[a^x b] * [b^y c] = [a^w c] = [a^z c]$. \square

In the works [3,4] there exists the notion of an elementary deformation of a path. They say that a path admits an *elementary deformation* if one can replace some section $[a^x b] * [b^y c]$ of the path with $[a^z c]$ and vice versa. It is possible in the conditions of the Corollary 1. If we can obtain a path $q \in S$ from some path $p \in S$ by a finite number of elementary deformations then according to the Lemma 2 and the Corollary 1 we have the equality $q = p$.

We say that $p \in S$ is a *loop* if $\partial_0 p = \partial_1 p$.

Let us denote by G_a the set of all loops that start and end in the point a .

Lemma 3. *The following statements hold:*

- 1) the set G_a is a subgroup in S with a unit i_a ;
- 2) each path p generates isomorphism between groups G_a and G_b if $\partial_0 p = a$, $\partial_1 p = b$;
- 3) if $p, q \in S$ and $\partial_0 p = \partial_0 q = a$, $\partial_1 p = \partial_1 q = b$, then there exist $g_1 \in G_a$ and $g_2 \in G_b$, such that $p = g_1 * q = q * g_2$.

Proof. 1) The first statement is obvious.

2) Define a map $\gamma_p : G_a \rightarrow G_b$ in the following way:

$$\gamma_p(g) = p^{-1}gp,$$

where $g \in G_a$. One can check that γ_p is an isomorphism.

3) It is easy to see that the statement holds for $g_1 = p * q^{-1} \in G_a$ and $g_2 = q^{-1} * p \in G_b$. \square

Lemma 4. *If the poset K is an upward directed set then the following statements hold:*

1) *for all $a, b, x, y \in K$ if $a, b \leq x$ and $a, b \leq y$ then*

$$[a^x b] = (a, x) * \overline{(x, b)} = (a, y) * \overline{(y, b)} = [a^y b];$$

for simplicity let us omit supports and denote a 1-simplex by $[a, b]$;

2) *$[a, b] * [b, c] = [a, c]$ for all $a, b, c \in K$;*

3) *for every $p \in S$ if $\partial_0 p = a$ and $\partial_1 p = b$ then $p = [a, b]$;*

4) *if $g \in G_a$ then $g = i_a$ and the group G_a is a trivial group.*

Proof. 1) As the poset K is upward directed set then there exists $z \in K$, such that $x, y \leq z$. Hence, we have

$$\begin{aligned} [a^x b] &= (a, x) * \overline{(x, b)} = (a, x) * (x, z) * \overline{(z, x)} * \overline{(x, b)} = (a, z) * \overline{(z, b)} = \\ &= (a, y) * (y, z) * \overline{(z, y)} * \overline{(y, b)} = (a, y) * \overline{(y, b)} = [a^y b]. \end{aligned}$$

2) It follows from Lemma 2.

3) It follows from 2).

4) For every $g \in G_a$ we have $g = [a, a_n] * \dots * [a_2, a_1] * [a_1, a]$. Using 2) several times, one gets $g = [a, a_1] * [a_1, a] = [a, a] = (a, a) * \overline{(a, a)} = i_a$. \square

3. C^* -ALGEBRA $C_{red}^*(S)$

In this section we define the C^* -algebra $C_{red}^*(S)$ generated by the representation of the path semigroup S and obtain the net of isomorphic C^* -algebras $(\mathcal{A}_a, \gamma_{ba}, a \leq b)_{a, b \in K}$ over the poset K , where $\gamma_{ba} : \mathcal{A}_a \rightarrow \mathcal{A}_b$ are $*$ -isomorphisms satisfying the identity $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for $a \leq b \leq c$.

Let us consider a Hilbert space

$$l^2(S) = \left\{ f : S \rightarrow \mathbb{C} \mid \sum_{p \in S} |f(p)|^2 < \infty \right\}$$

with an inner product $\langle f, g \rangle = \sum_{p \in S} f(p) \overline{g(p)}$. A family of functions $\{e_p\}_{p \in S}$ is an ortonormal basis of $l^2(S)$ where $e_p(p') = \delta_{p, p'}$ is a Kronecker symbol. Let $B(l^2(S))$ be the algebra of all linear bounded operators acting on $l^2(S)$.

Define a representation $\pi : S \rightarrow B(l^2(S))$ by $\pi(p) = T_p$ where

$$T_p e_q = \begin{cases} e_{p*q} & \text{if } \partial_1 p = \partial_0 q, \\ 0 & \text{otherwise.} \end{cases}$$

Note that π is the left regular representation and coincides with the Vagner representation of an inverse semigroup (see the definition of the Vagner representation in [14]).

We have $\langle T_p e_q, e_r \rangle \neq 0$ if and only if $p * q = r$ or $q = p^{-1} * r$. Hence,

$$\langle T_p e_q, e_r \rangle = \langle e_q, T_{p^{-1}} e_r \rangle.$$

Define the adjoint operator $T_p^* = T_{p^{-1}}$. In Lemma 5 we show that operators T_p and T_p^* are partial isometric operators.

Given $a \in K$ we define $S_a = \{p \in S \mid \partial_0 p = a\}$. Thus $l^2(S)$ can be written as

$$l^2(S) = \bigoplus_{a \in K} l^2(S_a).$$

Lemma 5. *The following statements hold:*

- 1) for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$, the operator T_p is a mapping from $l^2(S_b)$ to $l^2(S_a)$ and the operator T_p^* is an inverse mapping from $l^2(S_a)$ to $l^2(S_b)$;
- 2) for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$, operators $I_a = T_p T_p^*$ and $I_b = T_p^* T_p$ are projectors on $l^2(S_a)$ and $l^2(S_b)$ respectively;
- 3) for every $g \in G_a$ the operator T_g is a unitary operator on $l^2(S_a)$;
- 4) for all $p, q \in S$, such that $\partial_0 p = \partial_0 q = a$, $\partial_1 p = \partial_1 q = b$, there exist $g_1 \in G_a$ and $g_2 \in G_b$, such that $T_p = T_{g_1} T_q = T_q T_{g_2}$.

Proof. 1) We observe that $T_p e_q = e_{p*q}$ if $\partial_0 q = b$ and $T_p e_q = 0$ otherwise. Since $\partial_0(p * q) = a$ then $T_p : l^2(S_b) \rightarrow l^2(S_a)$. Similarly, $T_p^* : l^2(S_a) \rightarrow l^2(S_b)$.

2) It is easy to see that $I_a e_q = T_p T_p^* e_q = e_{p*p^{-1}*q} = e_q$ if $\partial_0 q = a$ and $I_a e_q = 0$ otherwise. Therefore, I_a is a projector on $l^2(S_a)$. Similarly, one can prove that I_b is a projector on $l^2(S_b)$.

3) We have $T_g : l^2(S_a) \rightarrow l^2(S_a)$ and $T_g T_g^* e_p = e_{g*g^{-1}*p} = e_p$, $T_g^* T_g e_p = e_p$ for every $p \in S_a$. Hence, T_g is a unitary operator.

4) This statement follows from the Lemma 3 (item 3). \square

Let us denote by $C_{red}^*(S)$ a uniformly closed subalgebra of $B(l^2(S))$ generated by operators T_p for every $p \in S$. Obviously the set of finite linear combinations of operators T_p , $p \in S$, is dense in $C_{red}^*(S)$.

Given $a \in K$ we denote $S^a = \{p \in S \mid \partial_1 p = a\}$. Thus we have again

$$l^2(S) = \bigoplus_{a \in K} l^2(S^a).$$

Theorem 2. *The following statements hold:*

- 1) *the algebra $C_{red}^*(S)$ is irreducible on $l^2(S^a)$ for every $a \in K$;*
- 2) *$C_{red}^*(S) = \bigoplus_{a \in K} C_{red}^*(S)|_{l^2(S^a)}$ and every operator $A \in C_{red}^*(S)$ can be represented as $A = \bigoplus_{a \in K} A_a$ where $A_a = A|_{l^2(S^a)}$;*
- 3) *if the group G_a is non-trivial then $C_{red}^*(S)|_{l^2(S^a)}$ doesn't contain compact operators.*

Proof. 1) The set $\{e_p, \partial_1 p = a\}_{p \in S}$ is a basis of $l^2(S^a)$. For all $p_1, p_2 \in S^a$ and $p = p_2 * p_1^{-1}$ we have $T_p e_{p_1} = e_{p_2}$. It means that the algebra $C_{red}^*(S)$ is irreducible on $l^2(S^a)$.

2) This statement follows from the fact that for every $p \in S$ operator T_p maps the space $l^2(S^a)$ onto itself for every $a \in K$.

3) Let $p \in S^a$, $g \in G_a$ and $g \neq i_a$. Consider the sequence $x_n = e_{p * g^n}$ where $g^n = \underbrace{g * g * \dots * g}_n$. Since $g * g \neq g$ elements of the sequence

$\{x_n\}$ are pairwise orthogonal. If $A \in C_{red}^*(S)|_{l^2(S^a)}$ is a compact operator then $\|Ax_n\| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand $Ae_p = \sum_i \alpha_i e_{p_i}$ where $p_i \in S^a$ and α_i are complex coefficients. Referral to the fact that A is approximated by finite linear combinations of operators T_q , $q \in S$, and to the equality $T_q e_{p * g} = e_{q * p * g}$ we obtain $Ae_{p * g} = \sum_i \alpha_i e_{p_i * g}$. Similarly $Ae_{p * g^n} = \sum_i \alpha_i e_{p_i * g^n}$ for all n . Therefore, for every n we have $\|Ax_n\| = \left(\sum_i |\alpha_i|^2 \right)^{1/2} > 0$. Hence, A is not a compact operator. \square

Theorem 3. *Let K be an upward directed set. Then the following statements hold:*

- 1) *for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$, we have $T_p = T_{[a,b]}$;*
- 2) *for every $a \in K$ the algebra $C_{red}^*(S)|_{l^2(S^a)}$ coincides with the algebra of all compact operators on $B(l^2(S^a))$;*
- 3) *the algebra $C_{red}^*(S)$ is non-unital.*

Proof. 1) This statement follows from the Lemma 4.

2) The set $\{e_{[c,a]}\}_{c \in K}$ is a basis of $l^2(S^a)$. For every operator T_p we have $T_p e_{[c,a]} \neq 0$ if and only if $\partial_1 p = c$. Hence, $T_p = T_{[b,c]}$ for some b and $T_{[b,c]} e_{[c,a]} = e_{[b,a]}$. Therefore, $T_p|_{l^2(S^a)}$ is a one dimensional operator. So C^* -algebra $C_{red}^*(S)|_{l^2(S^a)}$ coincides with the algebra of all compact operators on $B(l^2(S^a))$.

3) By the Theorem 2 for every element $A \in C_{red}^*(S)$ we have $A = \bigoplus_{a \in K} A_a$ where $A_a \in C_{red}^*(S)|_{l^2(S^a)}$. If the algebra $C_{red}^*(S)$ has the unit

I then $I_a = I|_{l^2(S^a)}$ is a compact operator in the infinite dimensional Hilbert space. This is a contradiction. \square

Given $a \in K$ we denote $\mathcal{A}_a = C_{red}^*(S)|_{l^2(S^a)}$.

Theorem 4. *There exists the set of $*$ -isomorphisms $\{\gamma_{ba}, a \leq b\}_{a,b \in K}$:*

$$\gamma_{ba} : \mathcal{A}_a \rightarrow \mathcal{A}_b,$$

such that $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for all $a, b, c \in K$ and $a \leq b \leq c$. And we obtain a net of isomorphic C^ -algebras $\{\mathcal{A}_a, \gamma_{ba}, a \leq b\}_{a,b \in K}$ over the poset K .*

Proof. Define a unitary operator $U_{ab} : l^2(S^a) \rightarrow l^2(S^b)$ for all $a, b \in K$, $a \leq b$, by

$$U_{ab}e_q = e_{q*(a,b)}$$

for every $q \in S^a$. Then $U_{ab}^* = U_{ba} : l^2(S^b) \rightarrow l^2(S^a)$ is the adjoint operator. Obviously, $U_{ab}^*U_{ab} = id|_{l^2(S^a)}$ and $U_{ab}U_{ab}^* = id|_{l^2(S^b)}$. Let us define a mapping $\gamma_{ba} : \mathcal{A}_a \rightarrow \mathcal{A}_b$ by

$$\gamma_{ba}(A) = U_{ab}AU_{ab}^*$$

for every $A \in \mathcal{A}_a$. One can check that γ_{ba} is the $*$ -isomorphism. It remains to check the equality $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for $a \leq b \leq c$. We observe that

$$(\gamma_{cb} \circ \gamma_{ba})(A) = \gamma_{cb}(\gamma_{ba}(A)) = U_{bc}U_{ab}AU_{ab}^*U_{bc}^*$$

for every $A \in \mathcal{A}_a$. Otherwise

$$U_{bc}U_{ab}e_q = U_{bc}e_{q*(a,b)} = e_{q*(a,b)*(b,c)} = e_{q*(a,c)} = U_{ac}e_q$$

for every $q \in S^a$ and similarly $U_{ab}^*U_{bc}^*e_p = U_{ac}^*e_p$ for every $p \in S^c$. So $(\gamma_{cb} \circ \gamma_{ba})(A) = \gamma_{ca}(A)$ for every $A \in \mathcal{A}_a$. \square

Remark 1. *The set of isomorphisms $\{\gamma_{ba}, a \leq b\}_{a,b \in K}$ can be extended from elementary paths to 1-simplices $\{\gamma_{[b^x a]}, a, b \leq x\}_{a,b,x \in K}$ by $\gamma_{[b^x a]} = \gamma_{xb}^{-1} \circ \gamma_{xa}$, so that they satisfy 1-cocycle identity [4]: $\gamma_{[c^y b]} \circ \gamma_{[b^x a]} = \gamma_{[c^z a]}$ for $[c^y b] * [b^x a] = [c^z a]$. Extending the set $\{\gamma_{[b^x a]}, a, b \leq x\}_{a,b,x \in K}$ to paths we get the set of isomorphisms $\{\gamma_p\}_{p \in S}$ satisfying the equality $\gamma_{p_2} \circ \gamma_{p_1} = \gamma_{p_2 * p_1}$ for all $p_1, p_2 \in S$ and $\partial_0 p_1 = \partial_1 p_2$.*

4. EXTENSIONS OF THE C^* -ALGEBRA $C_{red}^*(S)$

In this section we consider the extensions of the algebra $C_{red}^*(S)$, such that this algebra is an ideal in that extensions and quotient algebras are isomorphic to the Cuntz algebra.

Let K be an upward directed countable set. By the lemma 4 for every path $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$, we have $p = [a, b]$. Let us represent the set K as a finite union of countable disjoint sets

$$K = \bigcup_{i=1}^n E_i,$$

where $E_i \cap E_j = \emptyset$ for $i \neq j$.

We define one-to-one mappings $\phi_i : E_i \rightarrow K$, $i = 1, \dots, n$, and operators $T_{\phi_i} : l^2(S) \rightarrow l^2(S)$ in the following way:

$$T_{\phi_i} = \bigoplus_{x \in E_i} T_{[x, \phi_i(x)]}, \quad i = 1, \dots, n.$$

An adjoint operator of the operator T_{ϕ_i} is

$$T_{\phi_i}^* = \bigoplus_{x \in E_i} T_{[x, \phi_i(x)]}^* = \bigoplus_{x \in E_i} T_{[\phi_i(x), x]} = \bigoplus_{x \in K} T_{[x, \phi_i^{-1}(x)]}$$

The following equalities hold:

$$T_{\phi_i}^* T_{\phi_i} = id; \quad T_{\phi_i}^* T_{\phi_j} = 0, \quad i \neq j; \quad \sum_{i=1}^n T_{\phi_i} T_{\phi_i}^* = id.$$

Indeed every basis element has a form $e_{[a, b]}$. Therefore,

$$\begin{aligned} T_{\phi_i}^* T_{\phi_i} e_{[a, b]} &= T_{\phi_i}^* T_{[\phi_i^{-1}(a), a]} e_{[a, b]} = T_{\phi_i}^* e_{[\phi_i^{-1}(a), b]} = \\ &= T_{[a, \phi_i^{-1}(a)]} e_{[\phi_i^{-1}(a), b]} = e_{[a, b]}. \end{aligned}$$

Analogously, since $E_i \cap E_j = \emptyset$ we have $T_{\phi_i}^* T_{\phi_j} e_{[a, b]} = 0$. Finally if $a \in E_k$ then

$$\begin{aligned} \left(\sum_{i=1}^n T_{\phi_i} T_{\phi_i}^* \right) e_{[a, b]} &= T_{\phi_k} T_{[\phi_k(a), a]} e_{[a, b]} = T_{\phi_k} e_{[\phi_k(a), b]} = \\ &= T_{[a, \phi_k(a)]} e_{[\phi_k(a), b]} = e_{[a, b]}. \end{aligned}$$

Let us consider a uniformly closed subalgebra of $B(l^2(S))$ generated by operators T_p , $p \in S$, and T_{ϕ_i} , $i = 1, \dots, n$. Denote it by $C_{red, n}^*(S)$. The algebra $C_{red, n}^*(S)$ is unital. Hence, it doesn't coincide with $C_{red}^*(S)$. It is an extension of algebra $C_{red}^*(S)$. Moreover the following lemma holds.

Lemma 6. *The algebra $C_{red}^*(S)$ is an ideal in $C_{red, n}^*(S)$.*

Proof. We have $T_{\phi_i} T_{[a, b]} = T_{[x, b]}$ for some $x \in K$ and $T_{[a, b]} T_{\phi_i} = T_{[a, y]}$ for some $y \in K$. Since every element $A \in C_{red}^*(S)$ can be approximated by finite linear combinations of operators $T_{[a, b]}$ then $T_{\phi_i} A$ and $A T_{\phi_i} \in C_{red}^*(S)$. \square

Let us recall the definition of the Cuntz algebra. The *finite Cuntz algebra* O_n is a C^* -algebra generated by isometries s_1, \dots, s_n satisfying to the following conditions:

$$s_i^* s_j = \delta_{ij} id, \quad \sum_{i=1}^n s_i s_i^* = id.$$

The *infinite Cuntz algebra* O_∞ is a C^* -algebra generated by s_1, s_2, \dots and relations

$$s_i^* s_j = \delta_{ij} id, \quad \sum_{i=1}^n s_i s_i^* \leq id$$

for every $n \in \mathbb{N}$.

Theorem 5. *There exist an isomorphism $C_{red,n}^*(S)/C_{red}^*(S) \cong O_n$ and a short exact sequence*

$$0 \rightarrow C_{red}^*(S) \xrightarrow{id} C_{red,n}^*(S) \xrightarrow{\pi} O_n \rightarrow 0,$$

where id is an embedding map and π is a quotient map.

Proof. Equivalence classes $[T_{\phi_i}] = T_{\phi_i} + C_{red}^*(S)$, $i = 1, \dots, n$, are generators of the quotient algebra $C_{red,n}^*(S)/C_{red}^*(S)$. These classes are isometric operators satisfying the following identity:

$$\sum_{i=1}^n [T_{\phi_i}][T_{\phi_i}^*] = id.$$

Due to the universality of the Cuntz algebra we observe that

$$C_{red,n}^*(S)/C_{red}^*(S) \cong O_n.$$

□

Now let us represent the set K as a countable union of disjoint countable sets:

$$K = \bigcup_{i=1}^{\infty} E_i$$

and define operators $T_{\phi_i} : l^2(S) \rightarrow l^2(S)$ in the following way:

$$T_{\phi_i} = \bigoplus_{x \in E_i} T_{[x, \phi_i(x)]}, \quad i = 1, 2, \dots$$

By applying the reasoning used above one can prove the following equalities:

$$T_{\phi_i}^* T_{\phi_i} = id; \quad T_{\phi_i}^* T_{\phi_j} = 0, \quad i \neq j; \quad \sum_{i=1}^n T_{\phi_i} T_{\phi_i}^* \leq id$$

for every $n \in \mathbb{N}$.

Let us denote by $C_{red,\infty}^*(S)$ the uniformly closed subalgebra of $B(l^2(S))$ generated by operators T_p , $p \in S$, and T_{ϕ_i} , $i = 1, 2, \dots$

Similarly to the Lemma 6 the algebra $C_{red}^*(S)$ is an ideal in $C_{red,\infty}^*(S)$ and for the infinite Cuntz algebra the following theorem holds.

Theorem 6. *There exist an isomorphism $C_{red,\infty}^*(S)/C_{red}^*(S) \cong O_\infty$ and a short exact sequence*

$$0 \rightarrow C_{red}^*(S) \xrightarrow{id} C_{red,\infty}^*(S) \xrightarrow{\pi} O_\infty \rightarrow 0,$$

where id is an embedding map and π is a quotient map.

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REFERENCES

- [1] R. Haag, *Local quantum Physics: Fields, particles, algebras* (Springer-Verlag, Berlin, 1992).
- [2] G. Ruzzi, Homotopy of posets, net cohomology and superselection sectors in globally hyperbolic space-times, *Rev. Math. Phys.*, **17**, 1021-1070 (2005).
- [3] J.E. Roberts, More lectures in algebraic quantum field theory, in: S. Doplicher, R. Longo (Eds.), *Noncommutative Geometry. C.I.M.E. Lectures*, Martina Franca, Italy, 2000, Springer-Verlag (2003).
- [4] G. Ruzzi, E. Vasselli, A new light on nets of C*-algebras and their representations, *Comm. Math. Phys.*, **312**, 655-694 (2012).
- [5] E. Vasselli, Presheaves of symmetric tensor categories and nets of C*-algebras, *Journal of Noncommutative Geometry*, **9**, 121-159 (2015).
- [6] E. Vasselli, Presheaves of superselection structures in curved spacetimes, *Comm. Math. Phys.*, **335**, 895-933 (2015).
- [7] R. Brunetti, G. Ruzzi, Quantum charges and space-time topology: The emergence of new superselection sectors, *Comm. Math. Phys.*, **287**, 523-563 (2009).
- [8] M.A. Aukhadiev, S.A. Grigoryan, E.V. Lipacheva, Infinite-dimensional compact quantum semigroup, *Lobachevskii Journal of Mathematics*, **32** (4), 304-316 (2011).
- [9] M.A. Aukhadiev, V.H. Tepoyan, Isometric representations of totally ordered semigroups, *Lobachevskii Journal of Mathematics*, **33** (3), 239-243 (2012).
- [10] S.A. Grigoryan, V.H. Tepoyan, On isometric representations of the perforated semigroup, *Lobachevskii Journal of Mathematics*, **34** (1), 85-88 (2013).
- [11] T.A. Grigoryan, E.V. Lipacheva, V.H. Tepoyan, On the extension of the Toeplitz algebra, *Lobachevskii Journal of Mathematics*, **34** (4), 377-383 (2013).
- [12] M.A. Aukhadiev, A.S. Nikitin, A.S. Sitdikov, Crossed product of the canonical anticommutative relations algebra in the Cuntz algebra, *Russian Mathematics*, **58** (8), 71-73 (2014).
- [13] A.H. Clifford, G.B. Preston, *The algebraic theory of semigroups* V.1. (AMS, 1964).

- [14] Alan L.T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras* (Birkhauser, 1998).
- [15] V.V. Vagner, Generalized groups (Russian), Doklady Akad. Nauk SSSR **84**, 1119-1122 (1952).